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FRACTAL GEOMETRY IN SOLIDS AND STRUCTURES†

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Abstract—The present paper presents certain methods for the consideration of the influence of the fractal geometry in solid and structural mechanics. The method of fractal interpolation function is introduced and the fractals are considered as the fixed "points" of a given set-valued transformation. An attempt is made here to define the mechanical quantities on fractal sets using some results of the theory of Besov spaces. Then we try to extend the classical FEM for the case of fractal bodies and fractal boundaries. Then corresponding error estimates are derived. The methods of fractal analysis permit the treatment of complicated or yet unsolved problems in engineering.

1. INTRODUCTION

The geometry of fractals is very suitable for the accurate geometrical description of certain physical objects and of the figures and graphs resulting in physical and chemical processes. One could mention here the landscape and coastline geometry, the fluvial system geometry, the form of clouds and mountains, the distribution of craters in planets, the geometry of cracked and crushed interfaces (Fig. 1), the diffusion front geometry, the aggregation patterns, the percolation transition patterns etc. [see e.g. Mandelbrot (1972), Takayasu (1990), Scholz and Mandelbrot (1989), Le Méhauté (1990), Barnsley and Demko (1986) and Feder (1988)].

There are several nonrigorous mathematical approaches to the theory of fractals and a few rigorous mathematical ones [see Barnsley (1988), Falconer (1985) and Wallin (1989)]. The approach of Barnsley (1988) seems to be more suitable for the engineering sciences. Following here the definition of Mandelbrot (1972), Wallin (1989), we consider that a set $F \subset R^n$ is a fractal set if F has a fractional "Hausdorff" dimension, or if the dimension of F is an integer strictly larger than the topological dimension of F. It is more or less obvious that the classical mechanics are based on the inherent assumption of integral dimension and that the consideration of fractal geometry with a non-integer dimension leads to another type of mechanics: the "mechanics of fractals".

Until now connections between the theory of fractals and mechanics concerned only the manifestation of the fractal nature of some physical objects and the calculation of their "Hausdorff dimension", with special emphasis on the study of the attractors of some dynamic systems. The author (Panagiotopoulos, 1989, 1990a,b,c) has examined several other connections between the theory of fractals and mechanics which focused especially on the three following subjects:

(i) the definition of mechanical quantities on fractals;
(ii) the extension of structural analysis calculations to fractal geometries;
(iii) the fractal approximation and the solution of partial differential equations on a domain with fractal boundaries.

Indeed, in several physical phenomena we have fractal geometries or fractal surfaces and interfaces [*cf.* Feder (1989)], i.e. we are confronted with fractal domains Ω and/or boundaries Γ. This is the case in the phase transition problem (Takayashu, 1990) and in diffusion fronts (Le Méhauté, 1990); of fractal type is the geometry of a metal surface after it is subjected to a meteorite rain or after a sandblasting procedure [*cf.* Le Méhauté (1990)

† Dedicated to Professor George Herrmann on the occasion of his 70th birthday.

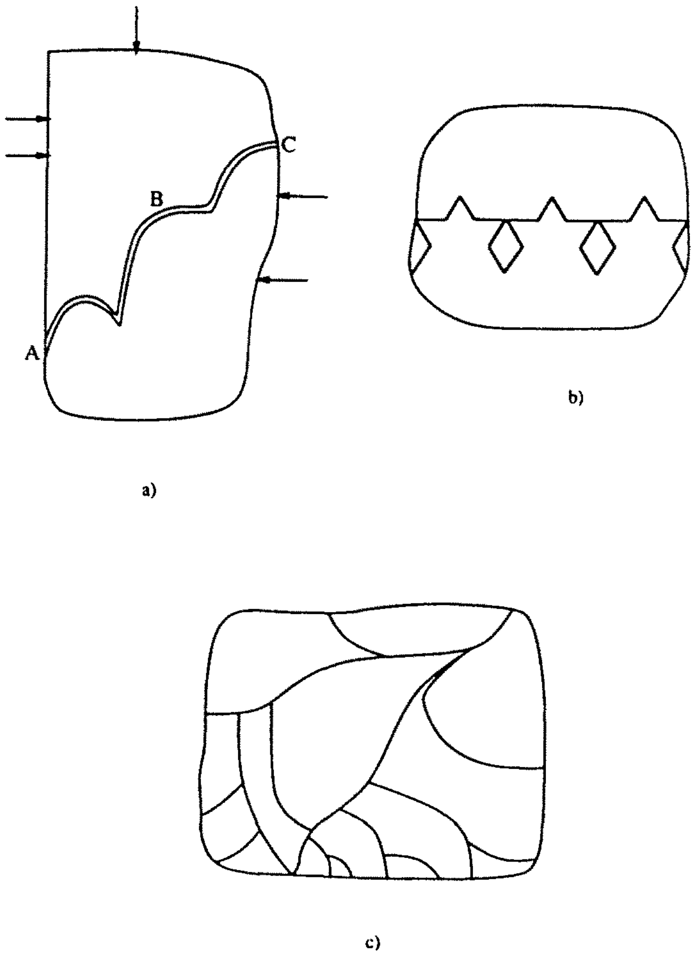


Fig. 1. Cracked interfaces of fractal geometry; in (b) the fractal dimension is $D = \log 4/\log 3$.

planche 7] and of fractal geometry are the cracks and the interfaces in solids and in composite structures [*cf.* Takayashu (1990), p. 78 and Saouma *et al.* (1990)].

In the present paper the influence of the domain and/or boundary fractal geometry on the displacement, stress, etc. fields of the body is investigated. Methods for the consideration of this fractal geometry in the FEM calculations are proposed. Here it is assumed that the fractal geometry of Ω and/or of Γ does not change during the loading process and that any type of boundary conditions, i.e. bilateral or unilateral, may hold on the fractal boundary. Note that fractal geometry has already been associated with cracks by several authors now [*cf.* e.g. Mandelbrot (1972), Takayashu (1990), p. 78] because of the obvious relationship between the size of the fracture mirror zone and of the possible crack bifurcations with the roughness and the topography of the interface.

Finally we should mention that every complicated geometry is not necessarily of fractal nature (Scholz and Mandelbrot, 1989, p. 241). Here, however, the applied method for the approximation of the fractal through classic curves is applicable not only to fractals but also to any kind of the more "simple" curves or surfaces of integer Hausdorff dimension.

2. ON THE APPROXIMATION OF THE FRACTAL GEOMETRY BY CLASSICAL CURVES AND SURFACE CURVES

A set $A \subset \mathbb{R}^n$ is called a fractal set if its "Hausdorff dimension" dim A is fractional or if it is an integer but greater than its topological dimension (Wallin, 1989). Here we shall not explain the mathematical term "Hausdorff dimension". The reader should understand it as a mathematical notion analogous to the classical dimension. For a rigorous mathematical definition we refer to Falconer (1985). Let $\{X, d\}$ be a complete metric space with the metric d. We denote by $H(X)$ the space of the compact subsets of X. If $d(A, B)$ is the distance between the sets $A \subset X$ and $B \subset X$ defined by the formula

$$d(A, B) = \max_{x \in A} \min_{x \in B} d(x, y). \tag{1}$$

Then $H(X)$ endowed with the metric

$$h(A, B) = \max \{d(A, B), d(B, A)\} \; \forall \; A, B \in H(X) \tag{2}$$

is a complete metric space which is called the space of fractals. An iterated function system (I.F.S.) on X consists of n contractive mappings w_i: $X \to X$ with contractivity factors $0 \leqslant s_i < 1$, $i = 1, \ldots, n$, i.e.

$$d(w_i(x), w_i(y)) \leqslant s_i d(x, y) \; \forall \; x, y \in X, \quad 0 \leqslant s_i < 1. \tag{3}$$

The following proposition holds (Barnsley, 1988):

Proposition. Let $\{X; w_i, i = 1, \ldots, n\}$ be an I.F.S. We define W_i: $H(X) \to H(X)$ by setting

$$W_i(B) = \{w_i(x); x \in B\} \; \forall \; B \in H(x) \tag{4}$$

and let

$$W(B) = W_1(B) \cup W_2(B) \cup \cdots \cup W_n(B) \; \forall \; B \in H(x). \tag{5}$$

Then W is a contraction mapping on $H(X)$ with contractivity factor $s = \max \{s_1, \ldots, s_n\}$. The unique fixed "point" of W is the set $A \subset H(X)$ such that

$$A = W(A) = \bigcup_{i=1}^{n} W_i(A) \tag{6}$$

and is given by the relation

$$A = \lim_{m \to \infty} W^{(m)}(B) \; \forall \; B \in H(X), \tag{7}$$

where

$$W^{(0)}(x) = x, \quad W^{(m)}(x) = W(W^{(m-1)}(X)), \quad m = 1, 2, \ldots \tag{8}$$

are the forward iterates of W.

The set A is called the "attractor" of the I.F.S. $\{X; w_i\}$ and is the deterministic fractal of the I.F.S. considered.

Conversely, suppose that $C \in H(X)$ is given and let $\varepsilon > 0$ be a given small number. Suppose that it is possible to define on I.F.S. $\{X; w_i \; i = 1, \ldots, n\}$ such that

$$h\left(C, \bigcup_{i=1}^{n} W_i(C)\right) \leqslant \varepsilon. \tag{9}$$

Then the Hausdorff distance between C and the attractor A of the I.F.S. can become smaller than ε [collage theorem, Barnsley (1988)].

An important tool in the theory of fractals which has already found and will find several important applications in the mechanics of structures is "fractal interpolation".

Suppose that on $\mathbb{R}^2$, for instance, we have a set of data $\{x_i, y_i\}$ $i = 1, \ldots, N$. We seek a fractal interpolation function y: $[x_0, x_N] \to \mathbb{R}$, i.e. a fractal function f such that

$y(x_i) = y_i\ i = 0, 1, \ldots, N$. We consider the I.F.S. $\{\mathbb{R}^2; w_n\ n = 1, \ldots, N\}$ defined by the "shear transformation"

$$(x, y) \to w_i(x, y) = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \quad i = 1, \ldots, N. \tag{10}$$

Let the factors d_i, called scaling factors, satisfy $0 \leqslant d_i < 1$; they are free parameters of the problem. Moreover we have the relation

$$a_i = \frac{(x_i - x_{i-1})}{(x_N - x_0)}, \quad e_i = \frac{(x_n x_{i-1} - x_0 x_i)}{(x_N - x_0)}, \quad c_i = \frac{(y_i - y_{i-1})}{(x_N - x_0)} - d_i \frac{(y_N - y_0)}{(x_N - x_0)} \tag{11, 12}$$

$$f_i = \frac{(x_N y_{i-1} - x_0 y_i)}{(x_N - x_0)} - d_i \frac{(x_N y_0 - y_N x_0)}{(x_N - x_0)}. \tag{13}$$

The following proposition holds (Barnsley, 1988).

Proposition. Let F be the attractor of the I.F.S. defined by (10)–(13). Then F is the graph of a continuous function y: $[x_0, x_N] \to \mathbb{R}$ interpolating the data $\{x_i, y_i\}, i = 0, 1, \ldots, N$. If C^0 is the set of all continuous functions y: $[x_0, x_N] \to \mathbb{R}$ then the sequence of functions $\tilde{y}_{m+1}(x) = (T\tilde{y}_m)(x)$, where the operator T: $C^0 \to C^0$ is defined by

$$T(\tilde{y}(a_i x + e_i)) = c_i x + d_i \tilde{y}(x) + f_i, \quad i = 1, 2, \ldots, N \tag{14}$$

converges to the attractor F as $m \to \infty$.

It can be easily shown (Barnsley, 1988) that if the points $x_0, \ldots, x_n$ are equally placed then $\dim F = D$ is given by the formula

$$D = 1 + \frac{\ln\left(\sum_{i=1}^{N} |d_i|\right)}{\ln N} \tag{15}$$

if the points (x_i, y_i) $i = 0, 1, \ldots, N$ do not constitute a straight line (in this case $D = 1$) and if $\sum_{i=1}^{N} |d_i| > 1$. Note that the proper choice of the parameters d_i may make D very close to 1, i.e. it is a line-like fractal, or very close to 2, i.e. it is a surface-like fractal [e.g. Barnsley (1988), p. 230].

Accordingly we can assume that a fractal F is the fixed "point" of a given non-linear transformation and it can be approximated, either combining collage theorem with the attractor property (7) or by using the fractal interpolation property, by a sequence of sets F_n of integer dimension.

3. STRENGTH AND DYNAMICS CALCULATIONS FOR BODIES WITH FRACTAL GEOMETRY

Let us assume that we want to extend the formulae of classical theory of strength of materials to bodies with a fractal boundary F under a given set of forces. We assume for the sake of simplicity that we are in $\mathbb{R}^2$ and that F is a fractal interpolation function of the data (x_i, y_i) $i = 1, 2, \ldots, N$, or that an I.F.S. exists for which F is the attractor. Accordingly F is the fixed point of an operator Q and thus it results as the limit of a sequence of non-fractal boundaries F_n as $n \to \infty$. Note that $F_{n+1} = QF_n$. The strength calculations and the motion of such a body are reduced to the calculation of certain integrals of the form

$$I = \int_{x_0}^{x_N} g(x, F(x))\,dx \tag{16}$$

where g is a continuous function. Let us assume first that $(x, z) \to g(x, z)$ is a polynomial of general type with respect to x and z. Then it is possible to calculate I as a function of a_i, c_i, d_i, f_i, e_i, if F is a fractal interpolation function. Let us assume, e.g. that we have to calculate $I = \int xF^2(x)\,dx$. Using the substitution $x = a_i\tilde{x} + e_i$ we have that

$$I = \int_{x_0}^{x_N} g(x, F(x)\,dx = \int_{x_0}^{x_N} g(x, Q(F(x)))\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (g(x, Q(F(x)))\,dx)$$

$$= \sum_{i=1}^{N} \int_{x_0}^{x_N} g(a_i\tilde{x} + e_i, c_i\tilde{x} + d_iF(\tilde{x}) + f_i)d(a_i\tilde{x} + e_i) \tag{17}$$

from which I results as a function of $\int F(x)$, $\int xF(x)$, $\int F^2(x)$. Then again using (17) these integrals are obtained as functions of a_i, c_i, d_i, f_i, e_i. In order to avoid the aforementioned recursive method we can apply a limit procedure which is valid for any type of function g and both for the case "F is a fractal interpolation function" and "F results from an I.F.S." Indeed from (17) we obtain by applying the formula $F_{n+1} = QF_n$ that

$$I_{n+1} = \int_{x_0}^{x_N} g(x, F_{n+1}(x))\,dx = \sum_{i=1}^{N} \int_{x_0}^{x_N} g(a_i\tilde{x} + e_i, c_i\tilde{x} + d_iF_n(\tilde{x}) + f_i)d(a_i\tilde{x} + e_i) \tag{18}$$

from which for n sufficiently large we get a quite good approximation of the value of the integral. Having calculated the necessary integrals we can apply the well known formulae of strength of materials and simple dynamics in order to have the necessary strength or dynamic behaviour results.

Example 1: The aforementioned method can be directly applied to the strength calculations for a rock mass without any form idealization, or for a metallic part (e.g. a delta wing) subjected to sandblasting procedure. In this case the integral in (17) represents the various moments of inertia and thus we can apply the exact calculation of I as a function of a_i, c_i, d_i, f_i, e_i, if F is a fractal interpolation function.

Example 2: Newtonian potentials for fractal bodies. In order to calculate the dynamics of bodies in a gravity field when these bodies have a fractal geometry (e.g. the surface of a planet) we need an extension of the forms of the classical Newton potentials, D_i, D_{ij}, D_{ijk} (Chadrasekhar, 1987) derived from the distributions ρ, ρx_i, $\rho x_i x_j$, $\rho x_i x_j x_k$ respectively by the formulae

$$D_0(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}', \quad D_i(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')x_i'}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}', \quad D_{ij} = G\int_{V'} \frac{\rho(\mathbf{x}')x_i'x_j'}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}'. \tag{19}$$

Here $\mathbf{x} = \{x_i\}$, $\mathbf{x} = \{x_i'\}$, $i = 1, 2, 3$ are vectors in $\mathbb{R}^3$ referred to an orthogonal Cartesian coordinate system, G is the gravitation constant and $\rho(\mathbf{x}')$ is the mass distribution at $\mathbf{x}'$. Then if the body V has fractal geometry we shall have instead of (19) the formulae

$$D_0(\mathbf{x}) = \lim_{n\to\infty} G\int_V \frac{\rho(\mathbf{x}_n')d\mathbf{x}_n'}{|\mathbf{x}-\mathbf{x}_n'|} D_i(\mathbf{x}) = \lim_{n\to\infty} G\int_{V_n} \frac{\rho(\mathbf{x}_n')x_{in}'}{|\mathbf{x}-\mathbf{x}_n'|}\,d\mathbf{x}_n' \quad \text{etc.} \tag{20}$$

which define the corresponding Newtonian potentials for fractal geometry if the corresponding limits exist. Here the sequences $\{\mathbf{x}_n'\}$ and $\{V_n\}$ approximate $\{\mathbf{x}'\}$ and $\{V\}$ for the fractal body.

Example 3: Gaseous masses. The aforementioned results can be applied for the treatment of the non-radial oscillations of gaseous masses having fractal geometry. This is necessary in gaseous masses in astrophysical problems. Then the results of Chadrasekhar and Lebowitz (1964) combined with the limit procedure indicated here for the treatment of the fractal geometry offer the solution to the problem. It is interesting to note that in the case of a fractal geometry the virial equations of various orders (Chadresekhar, 1987) with respect to the fractal F hold as the limits of sequences of virial equations of the same orders with respect to the classical geometry of the approximants F_n.

4. DEFINITION OF MECHANICAL QUANTITIES AND MECHANICAL LAWS ON FRACTALS. FRACTAL BOUNDARIES AND SPONGED BODIES

Very close to the previous topics is the problem of defining the mechanical quantities on fractal bodies. For instance, which is the meaning of the deformation tensor $\varepsilon = \{\varepsilon_{ij}\}$ in a deformable body $V \subset \mathbb{R}^3$ of fractal geometry? Again we can assume that there is a sequence of deformable bodies $V_n \subset \mathbb{R}^3$ having classical geometry such that $V_n \to V$ in the sense described in the previous section.

We denote further by A the physical quantity we want to define on V, for instance the tensor ε, etc. and we define A as the limit of A_n, as $n \to \infty$, where A_n are the same physical quantities defined on V_n, if this limit exists. This last condition plays in some cases an important role, because the limit may not exist: for instance, let us assume that we want to define the boundary traction $S_i = \sigma_{ij} n_j$ of a body $\Omega \subset \mathbb{R}^2$ having the stress tensor σ_{ij}. Moreover let us assume that the boundary Γ of the body is partly the graph of the Weierstrass function

$$x \to f(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k x, \quad \lambda > 1, \quad 1 < s < 2 \tag{21}$$

where $s \in (1, 2)$ and $\lambda > 1$, and which has a dimension $D \leqslant s$, i.e. it is a fractal set (Falconer, 1985). This function is everywhere continuous but nowhere differentiable. Thus we cannot define the traction on the fractal part of the plate boundary using the same tools as in classic mechanics. However, physical evidence shows that even in this case the definition of the traction should be possible. Analogous is the situation in the case of an elastic plate with a fractal boundary; we cannot define, at least in the usual sense, the boundary rotation $\partial z/\partial n$, if the boundary is nowhere differentiable. Suppose further that we want to define a mechanical law f between the quantities a and b on a fractal body V. The problem of redefining the mechanical laws for a fractal geometry seems to be a very complicated one if one considers that until now all physical theories have been based on the basic inherent assumption of integer dimensionality. However, using the limit procedure we can make a first heuristic attempt towards defining the mechanical law on V. We formulate first on V_n the law $f(a_n, b_n, V_n) = 0$, $n = 1, 2, \ldots$, where a_n and b_n approximate a and b as previously indicated. Then by definition, we define the limit

$$\lim_{n \to \infty} f(a_n, b_n, V_n) = 0 \tag{22}$$

as the mechanical law on V, provided that this limit exist. As an example we consider the classic Hooke's law in a fractal body. We write it in the form

$$\lim_{n \to \infty} (\{\sigma_n\} - \{C_n \varepsilon_n\}) = 0, \tag{23}$$

where C_n is Hooke's tensor of the V_n-body, if this limit exists. This is obvious if $\{C_n\}$ is the same in every V_n-body. However this is not always possible in the general case of non-linear constitutive laws.

Until now we have given some heuristic answers to the problem of defining mechanical quantities and laws on fractal sets. In order to give more complete answers a heavier mathematical apparatus is necessary: this is the apparatus of Besov spaces, as has been shown by Jonson and Wallin (1984) and by Wallin (1989a). Here we shall only give some results of their theory and we shall apply it in order to define mechanical quantities and laws on bodies with fractal geometry.

Let Ω be an open subset of $\mathbb{R}^n$ with the boundary Γ. We assume that Γ is a fractal with dimension $n-1 < d < n$. Let $W_k^p(\Omega)$ be the Sobolev space of the $L^p(\Omega)$ functions with distributional derivatives D^α up to order k in $L^p(\Omega)$, i.e. the space

$$W_k^p(\Omega) = \{u | u \in L^p(\Omega), \quad D^\alpha u \in L^p(\Omega) \quad |\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant k\} \tag{24}$$

equipped with the norm

$$\|u\|_{p,k} = \left\{ \sum_{0 \leqslant |\alpha| \leqslant k} \|D^\alpha u\|_{L^p}^p \right\}^{1/p}. \tag{25}$$

We say that $u \in L^1(\Omega)$ can be defined strictly at $x \in \Omega \cup \Gamma$ if the limit

$$\tilde{u}(x) = \lim_{r \to \infty} \frac{1}{m(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega} u(y) \, d\Omega \tag{26}$$

exists. Here $B(x,r)$ denotes a ball in R^n, i.e. the set

$$\left\{ x | x = \{x_1, \ldots, x_n\}, \sum_{i=1}^n x_i^2 \leqslant r^2 \right\},$$

and $m(B(x,r) \cap \Omega$ the area of the intersection of this ball with Ω. The trace of u to Γ is the function $u|_\Gamma$ defined by the formula

$$u|_\Gamma(x) = \tilde{u}(x) \tag{27}$$

at every $x \in \Gamma$ where $\tilde{u}(x)$ exists. In classic linear elasticity it is well known that the displacement field u_i is considered as an element $H^1(\Omega)$, $i = 1, 2, \ldots, n$ [H^1 is the Sobolev space $W^{1,2}(\Omega)$] and that the Sobolev space $H^{1/2}(\Gamma)$ is the natural space for the boundary displacements. Then the dual space of $H^{1/2}(\Gamma)$, the space $H^{-1/2}(\Gamma)$, is the space for the boundary tractions $S_i = \sigma_{ij} n_j$, $ij = 1, \ldots, n$. If Γ is a fractal of dimension d, then we obtain that a displacement field $u_i \in H^1(\Omega)$ no longer has a trace $u_i|_\Gamma$ in $H^{1/2}(\Gamma)$ but in the Besov space $B_\beta^{2,2}(\Gamma)$ where $\beta = 1 - (n - d/2)$. Indeed as shown by Wallin (1989) the trace operator $\mathrm{Tr} = u \to u|_\Gamma$ is for Γ fractal under some quite general conditions such that

$$\mathrm{Tr}: u \in W_k^p(\Omega) \to u|_\Gamma \in B_\beta^{p,p}(\Gamma), \tag{28}$$

where

$$\beta = k - \frac{n-d}{p}. \tag{29}$$

The operator Tr is a bounded linear surjective operator with a bounded linear right inverse. A function v is an element of $B_\alpha^{p,p}(\Gamma)$, $\alpha > 0$, $1 \leqslant p, q \leqslant \infty$, if $v \in L^p(\Gamma)$ and if for some sequence $\{c_v\}$ $v = 0, \ldots, \infty$ of positive numbers such that $(\Sigma c_v^q)^{1/2} < \infty$ we have

$$\|v - s(\pi)\|_{L^p(\Gamma)} \leqslant c_v 2^{-v\alpha} \tag{30}$$

where $s(\pi)$ are appropriately defined splines of degree equal to the integer part of α [e.g. Jonson and Wallin (1984), p. 135]. Then the norm of v in $B_\alpha^{p,q}$ is equal to

$$\|v\| = \|v\|_{L^p(\Gamma)} + \inf\,(\Sigma c_v^q)^{1/q} \tag{31}$$

where the infinum is taken over all possible sequences $\{c_v\}$.

From the above results it is easily obtained that in an elastic body $\Omega \subset \mathbb{R}^3$ the boundary displacements

$$u_i \in B_\beta^{2,2}(\Gamma) \quad \text{with} \quad \beta = 1 - \frac{3-d}{2} = \frac{d-1}{2} \tag{32}$$

and the boundary tractions S_i belong to the dual space $[B_\beta^{2,2}(\Gamma)]'$ if Γ has a dimension $2 < d < 3$. Analogously for a plate $\Omega \subset \mathbb{R}^2$ with Γ having $1 < d < 2$ the boundary displacement

$$u_i \in B_\beta^{2,2}(\Gamma) \quad \text{with} \quad \beta = 2 - \frac{2-d}{2} = \frac{d+2}{2} \tag{33}$$

and the shearing forces $\tilde{Q}$ at the boundary belong to the dual space. For the boundary rotations β is given by (32). Note that for $d = 2$ in the elastic body (resp. $d = 1$ in the plate), e.g. in the case of a classical piecewise smooth boundary, we obtain the classical space $H^{1/2}(\Gamma)$ [resp. $H^{3/2}(\Gamma)$]. Note that the functional framework of Besov spaces permits the appearance of stress concentrations blowing up to infinite at certain boundary points.

The above results which are based on the mathematical theory of Wallin give a precise answer to the question of how to define a mechanical quantity on a fractal boundary. Analogous are the mathematical questions related with the definition of mechanical quantities on a fractal set Ω of dimension d. This is the case for instance for spongy bodies. Suppose that $\Omega \subset \mathbb{R}^n$ and that Ω is a d-set $0 < d < n$. We say that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is strictly defined at $x \in \Omega$ if the limit

$$\bar{f}(x) = \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, \mathrm{d}\Omega \tag{34}$$

exists. Then we define the trace $f|_\Omega$ of f to Ω as the function given by

$$f|_\Omega(x) = \bar{f}(x) \tag{35}$$

at every x where $\bar{f}(x)$ exists. At every other point $f|_\Omega$ cannot be defined. It is shown by Jonson and Wallin (1984) that the trace operator $\mathrm{Tr}: f \to f|_\Omega$ is for Ω fractal (under some general conditions) such that

$$\mathrm{Tr}: f \in W_k^p(\mathbb{R}^n) \to B_\beta^{p,p}(\Omega) \tag{36}$$

where

$$\beta = k - \frac{(n-d)}{p} > 0. \tag{37}$$

The operator Tr is a bounded linear surjective operator with a bounded linear right inverse. The last results can be applied to the definition of the strain tensor in the case of a spongy body $\Omega \subset \mathbb{R}^3$ with fractal dimension $2 < d < 3$. More precisely relations (34) and (35) may be used for the definition of the strain tensor component $e_{ij}|_\Omega$ from $e_{ij} \in L^2(\mathbb{R}^3)$. However, in this case we cannot determine a Besov space for the strain tensor according to (36): Indeed $L^2(\mathbb{R}^n) \equiv W_0^2(\mathbb{R}^n)$ and thus (37) implies that $\beta < 0$.

Analogously the stress tensor is defined. It is apparent that many questions remain still open. However, a general result is that a law connecting certain mechanical quantities must be compatible with the structure of the functional spaces of the quantities, i.e. the mechanical laws seem to depend on the fractal dimension. But this last assertion is incompatible with the meaning of a mechanical law. Therefore the form of a mechanical law should imply that this law does not depend on the fractal dimensions of the objects on which this mechanical law applies. This new "postulate" is verified, e.g. in the case of Hooke's law in linear elasticity.

5. ANALYSIS OF FRACTAL STRUCTURES. THE METHOD OF APPROXIMATION OF FRACTAL

As we have mentioned, there are several cases in which a theory is needed for the stress and strain calculation of structures having fractal geometry. We mention, among others, the fractal geometry networks, with all possible applications in hydrology and biomechanics, the spongy materials, the thin plates whose surface is a Weierstrass-type function, e.g. in the case of metal sheets subjected to sandblast [*cf.* Le Méhauté (1990), planche 17, pp. 184, 185], and the case of fractured bodies with fractal boundaries or fractal interfaces. First we shall study continuous structures.

5.1. *Continuous structures*

Let us denote by Ω the domain of a continuous structure (e.g. a plate in bending etc.) and let Γ be its boundary. We assume that Ω and Γ are fractals and let $\Omega_j \to \Omega$ and $\Gamma_j \to \Gamma$ as $j \to \infty$ in the Hausdorff metric. First we consider the classic deformation problems with respect to the structure having instead of Ω and Γ, Ω_j and Γ_j, $j = 1,2,\ldots$ Let $X_j = \{\sigma_j, u_j\}$ be the corresponding solution. Then $\lim X_j$, as $j \to \infty$, is by definition the solution of the same problem with respect to the fractal Ω and Γ. We denote further by G the pair $[\Omega, \Gamma]$ and by G_j the pair $\{\Omega_j, \Gamma_j\}$.

Assume that X_j is given for every j by the solution of the equation

$$L(G_j)X_j = p(G_j) \tag{38}$$

where $p(G_j)$ is the loading of structure or any other factor, like temperature distribution etc., and $L(G_j)$ is an appropriate operator from the admissible space V into the same space V which is endowed with the inner product $(.,.)$ and norm $\|.\|$. (i.e. V is a Hilbert space). The following result may be proved:

Proposition 1. Suppose that $L(G_j): V \to V$ is a linear bounded and symmetric operator with the properties

(i) For every j

$$(L(G_j)X, X) \geq c\|X\|^2 \ \forall\, X \in V \tag{39}$$

where c is a constant independent of j.

(ii) For $j \to \infty$

$$\|L(G_j)X^* - L(G)X^*\| \to 0 \ \forall\, X^* \in V \tag{40}$$

and

$$\|p(G_j) - p(G)\| \to 0 \quad \text{in} \quad V. \tag{41}$$

Then $\|X_j \to X\| \to 0$ in V. Moreover X is solution of the problem $L(G)X = p$.

Proof. We have

$$c\|X_j\|^2 \leqslant (L(G_j)X_j, X_j) = (p(G_j), X_j) \leqslant \|p(G_j)\| \, \|X_j\| \tag{42}$$

and thus $\|X_j\| < c$. Accordingly $X_j \to \tilde{X}$ weakly in V. Now we have, due to the symmetry of $L(G_j)$ and $L(G)$,

$$(p(G_j), X^*) = (L(G_j)X^*, X_j) \to (L(G)X^*, \tilde{X}) = (L(G)\tilde{X}, X^*). \tag{43}$$

From (43) and (41) we obtain that $\tilde{X}$ is a solution of the problem

$$L(G)X = p. \tag{44}$$

Moreover $\tilde{X} = X$ because of the uniqueness of the solution of the problem (44). The uniqueness results easily from the property (39) for the operator $L(G)$ which is a direct consequence of (39) and (40). The estimate

$$\begin{aligned} c\|X_j - X\|^2 &\leqslant (L(G_j)(X_j - X), X_j - X) = (L(G_j)X_j, X_j) \\ &+ (L(G_j)X, X) - 2(L(G_j)X_j, X) = (p(G_j), X_j) - (p(G_j), X) + (L(G_j)X, X) \\ &- (p(G_j), X) \leqslant \|p(G_j)\| \, \|X_j - X\| + (L(G_j)X, X) - (p(G_j), X) \end{aligned} \tag{45}$$

implies for $j \to \infty$ with (40) and (41) the property (41) q.e.d.

This proposition justifies the replacement of the fractal geometric elements by a sequence of classical geometric elements for the calculation of structures having fractal geometry. By Panagiotopoulos (1989) another approximation result is proved with respect to structural analysis problems formulated as "fixed point problems", and as boundary integral equation problems.

5.2. *Discrete structures. Fractal networks*

Let us assume here that we have a linear elastic discrete structure, i.e. a frame or a truss, which in the general case is represented by a network. Our main assumption here is that the network D has fractal geometry and that it is approximated by classic networks D_n as $n \to \infty$ in the sense of the Hausdorff metric.

Networks with directed branches are considered. The nodes are denoted by Latin and the branches by Greek letters. We suppose that we have m nodes and v branches. We take as branch variables the "stress vector" s_γ and the "strain vector" e_γ. As node variables the applied force vector p_k and the displacement vector u_k are considered. In any branch an "initial strain" e_γ^0 may exist. The above given quantities are assembled in vectors $e, s, u, p, p \in \mathbb{R}^q$. The equilibrium matrix G permits us to write the following relations

$$Gs = p, \qquad e = G^{\mathrm{T}}u. \tag{46, 47}$$

Index T denotes the transpose of a matrix or a vector. The network analysis is completed physically by defining an algebraic structure on the network, consisting of the relation between the "stress" s_γ and the "strain" e_γ. We accept that s_γ is a linear function of the e_γ expressed in the form

$$s_\gamma = K_{o\gamma}(e_\gamma - e_\gamma^0), \quad \gamma = 1, \ldots, v. \tag{48}$$

Here $K_{o\gamma} = F_{o\gamma}^{-1}$ is the flexibility matrix. Thus for the whole structure we may write

$$s = \mathrm{diag}\{K_{o\gamma}(e_\gamma - e_\gamma^0)\}. \tag{49}$$

The problem to be solved consists of the determination of s, e, u, for given p and e_0. From the relations (46), (47) and (48) we are led to the following problem: Find $u \in \mathbb{R}^q$ so as to satisfy the equation

$$Ku = p. \tag{50}$$

Here $K_o = \text{diag}\{K_{01}, \ldots, K_{ov}\}$, and $K = G^T K_o G$. Now we assume that the network is a fractal one and we denote it by D. Since D is the limit of the sequence $\{D_n\}$ of classical networks we may write for each of them an equality analogous to (50), i.e. for a finite n, and then we can pass to the limit as $n \to \infty$. The convergence proof is analogous to the one of the previous proposition with some differences: Indeed the fractal network D usually has an infinite number of nodes and branches. Therefore instead of $\mathbb{R}^q$ we shall have for D an equality holding in an appropriately defined subspace of the space of infinite sequences l_2 [*cf.* Smirnov (1973), p. 547 and set $A \equiv K$]. The following proposition holds (we denote by $\|.\|$ the l_2-norm):

Proposition 2. Suppose that as $n \to \infty$

$$\|p(D_n) - p(D)\| \to 0 \quad \text{in} \quad l_2 \tag{51}$$

and that each network is appropriately constrained so that rigid-body displacements cannot occur in any network. Then the solution $u(D_n)$ tends to the solution $u(D)$ of the fractal network in the norm of the l_2-space.

The proof uses the technique used for the proof of the previous proposition and the results on the approximation of infinite matrices of Smirnov (1973), p. 548, and is omitted here.

6. THE F.E.M. ON FRACTAL SETS. FIRST ATTEMPTS

In this section we use the simple mathematical approach to the theory of the F.E.M. as it is formulated in Johnson (1987). Let V be a separate Hilbert space with the inner product $(.,.)$ and let $a(.,.)$ be a symmetric, continuous bilinear form which is V-elliptic, i.e.

$$a(u,u) \geq c\|u\|_V^2 \;\forall\, u \in V, \quad c \text{ constant} > 0. \tag{52}$$

The space V is defined on an open domain $\Omega \subset \mathbb{R}^n$. Let also l be a linear form on V. We deal with the variational equality

$$u \in V, \quad a(u,v) = (l,v) \;\forall\, v \in V \tag{53}$$

which has, due to (52), a unique solution. Moreover, let $F \subset \mathbb{R}^n$ be a fractal domain which is imbedded into Ω. We denote by V_h the finite dimensional subspace of V of dimension M and let $\varphi_1, \varphi_2, \ldots, \varphi_M$ a base of V_h. Then the discretized form of (52) reads

$$u_h \in V_h \; a(u_h, v) = (l,v) \;\forall\, v \in V_h. \tag{54}$$

From (54) we obtain the matrix equation $A\xi = b$ where $\xi = \{\xi_i\} \in \mathbb{R}^M$, $b = \{b_i\} \in \mathbb{R}^M$ with $b_i = l(\varphi_i)$ and $A = \{a_{ij}\}$ with $a_{ij} = a(\varphi_i, \varphi_j\}$ $i,j = 1, \ldots, M$ being the stiffness matrix.

Until now we have assumed that $F \subset \Omega$ and we have applied the classic F.E.M. to Ω. Now we will consider the restriction of V_h to F, i.e. the space $\tilde{V}_h = V_h/F$ and then the new discretized problem

$$u_h \in \tilde{V}_h \; a(u_h, v) = (l,v) \;\forall\, v \in \tilde{V}_h \tag{55}$$

which leads to a matrix equation analogous to the previous one with the difference that we must take into account the restriction of V_h to F. According to Johnson (1987) (53) and (54) imply the error estimate

$$\|u-u_h\|_V \leqslant \alpha\|u-v\|_V, \forall v\in V_h, \quad \alpha \text{ constant} > 0. \tag{56}$$

In order to extend this estimate to fractal problems we assume that

$$\|u-v\|_V \leqslant c\|\tilde{u}-\tilde{v}\|_{V/F}, \forall v\in V_h, \quad c \text{ constant} > 0 \tag{57}$$

where $\tilde{u}, \tilde{v}$ are the corresponding restrictions to F. For instance (57) holds for $V = H^1(\Omega)$. Let us give an example for a plane elastic body: let $\Omega \subset \mathbb{R}^2$ be a domain with the polygonal boundary Γ and let $F \subset \Omega$.

On Ω the differential equation

$$Au = f \quad \text{on} \quad F, \tag{58}$$

is considered with

$$u = 0 \quad \text{on} \quad \Gamma_1 \tag{59}$$

where Γ_1 is the boundary of F and $\Gamma \cap \Gamma_1$ contains the vertices of the polygonal line. Then Ω is subdivided into triagonal finite elements $K_1, \ldots, K_m$ and

$$V_h = \{v | v/K_i \quad \text{linear on} \quad K_i, \quad i = 1, \ldots, m, \quad v = 0 \quad \text{on} \quad \Gamma\}. \tag{60}$$

It is well known that $V = [\overset{o}{H}{}^1(\Omega)]^2$, in the case of elasticity. If the base functions have a value equal to 1 on each node N and the value zero on the adjacent triangle vertices and on the remaining triangles, then the matrix equation $A\xi = b$ corresponding to (54) has the form

$$a_{ij} = \int_{\bigcup_{i=1}^{m} K_i} a(\varphi_i, \varphi_j)\,\mathrm{d}x, \quad b_i = \int_{\bigcup_{i=1}^{m} K_i} f\varphi_i\,\mathrm{d}x, \tag{61}$$

whereas the matrix equation corresponding to (55) becomes

$$\tilde{a}_{ij} = \int_{\bigcup_{i=1}^{m} K_i\cap F} a(\varphi_i, \varphi_j)\,\mathrm{d}x, \quad \tilde{b}_i = \int_{\bigcup_{i=1}^{m} K_i\cap F} f\varphi_i\,\mathrm{d}x, \tag{62}$$

i.e. the integrals have to be calculated over the fractal sets $\bigcup^{m} K_i \cap F$. The method applied is the usual one: F is approximated by a sequence $\{F_n\}$ and thus we can have an appropriately good estimate of $\tilde{a}_{ij}$ and $\tilde{b}_i$.

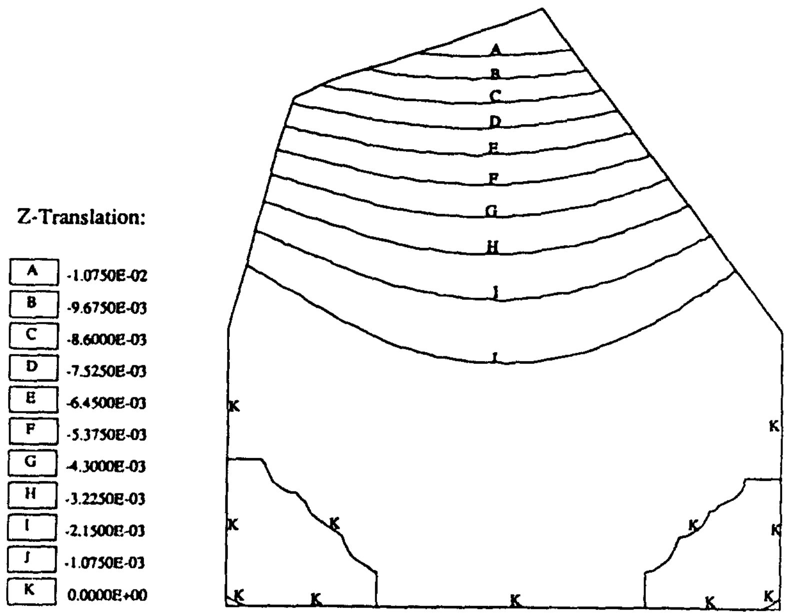
There is also another method for formulating the F.E. problem. Let us define the space V on F and let us formulate the corresponding discrete problem. Concerning the definition of function spaces on fractal sets we refer to Jonson and Wallin (1984). Then the error estimate (56) holds automatically. In order to continue we pose in (55), $v = \pi u$ where πu is an interpolant function of u and for which the estimates $\|u-\pi u\|$ are known in the case of classic geometry. For F fractal, in order to find the interpolation errors $\|u-\pi u\|$ we use the results of Wallin (1989). The following inequality holds (B is a ball with its center at x and with radius $r \leqslant 1$) under certain mild assumptions (preservation of Markov's inequality and k-unisolvability)

$$\sup_{F\cap B} |(v-\pi v)(x)| \leqslant cr^{k+1}\|v\|_{C^{k+1}(F\cap B)}. \tag{63}$$

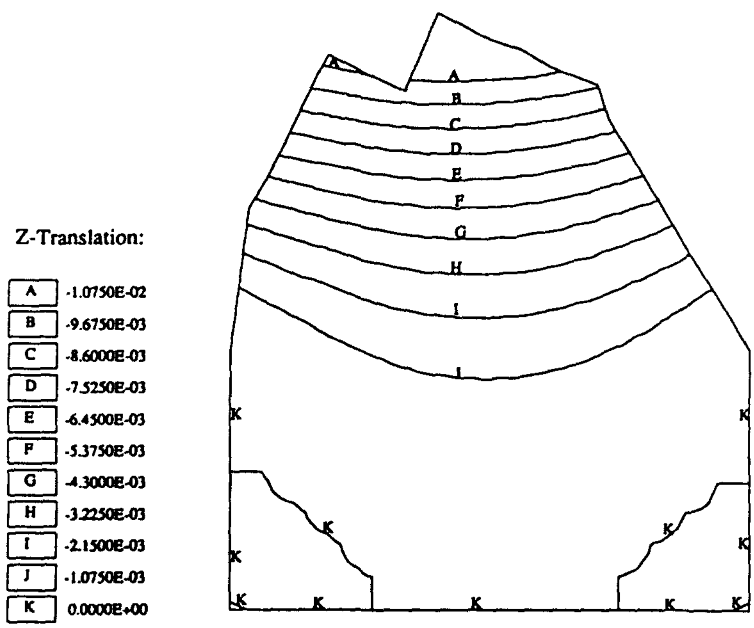
If more generally D_j is a partial derivative of jth order, the inequality

$$\sup_{F\cap B} |(D_j v - D_j(\pi v)(x))| \leqslant c r^{k+1-|j|} \| D_j u \|_{C^{k+1}_{|j| \leqslant k+1}(F\cap B)} \quad (64)$$

holds, where $0 \leqslant |j| \leqslant k$, $c = c(F, k, |j|, \Sigma)$ and Σ is the set of interpolation points. The spaces $C^{k+1}(F)$ of continuous functions where F is a fractal set are defined in Jonson and Wallin (1984). For the example defined by (58)–(59) the inequality (64) holds for $j = 0, 1$ for any triangulation satisfying the assumptions of Wallin (1989). The above results also give a partial answer to the question of the regularity of the solution of the F.E. scheme considered.

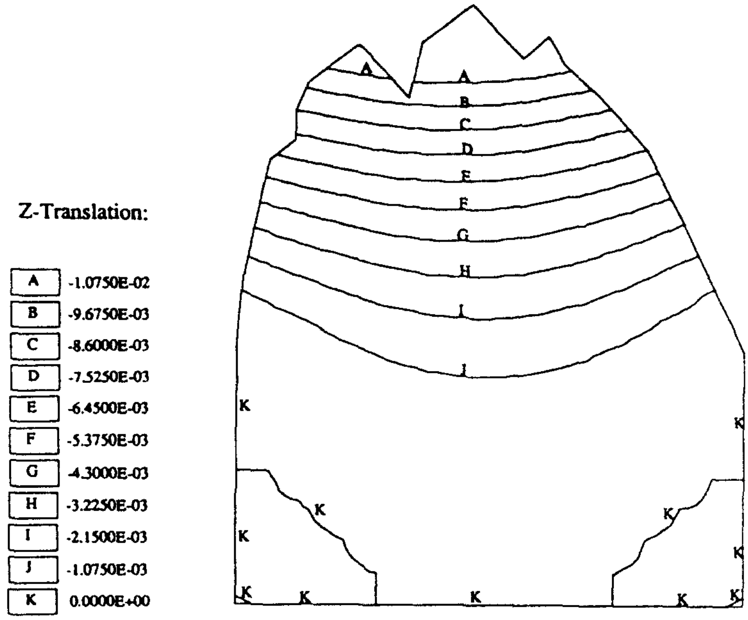


a)

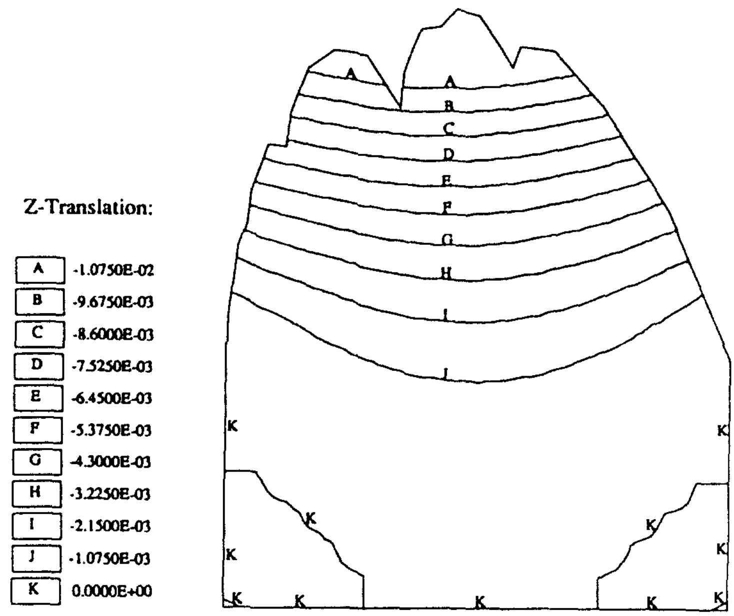


b)

Fig. 2. Calculation of the deflexions of the plate (*continued overleaf*).



c)



d)

Fig. 2 *continued*

7. APPLICATIONS

(i) As a first application we consider a Kirchhoff plate which in its undeformed state occupies a bounded domain Ω with a boundary Γ. The boundary is assumed to have fractal geometry. Moreover let Ω_j and Γ_j tend to Ω and Γ in the Hausdorff sense. Let us assume that mes Ω and mes Γ are finite numbers. The plate is simply supported on Γ. We formulate the plate problem in the space $H^2(\Omega_j)$ and noting the continuous injection $H^2(\Omega_j) \subset C^0(\Omega_j)$ we may easily deduce that (39) and (40) may be verified. Accordingly if all the plates Ω_j, $j = 1, 2, \ldots$, have the same loading, which does not change with the boundary Γ_j, then proposition 1 holds and at the limit we get the solution of the plate with the fractal boundary.

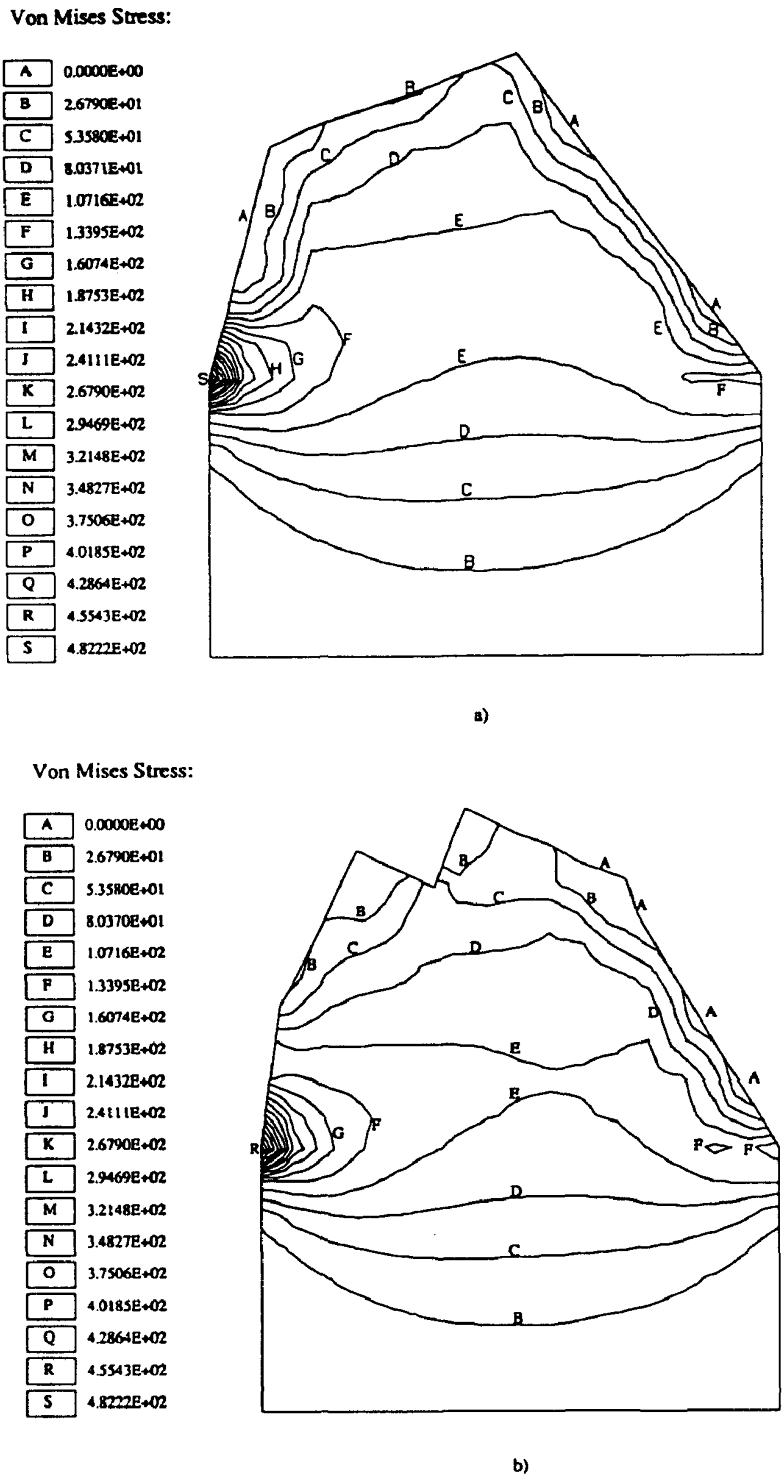
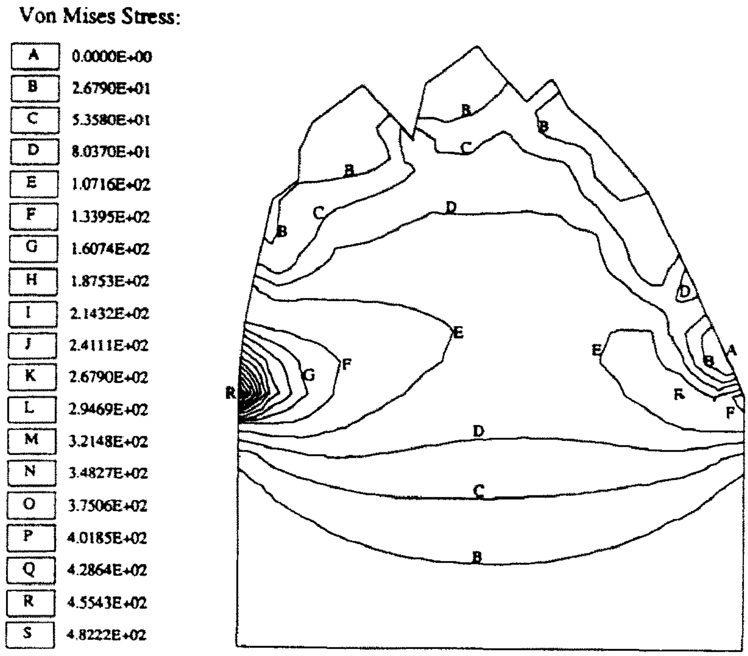


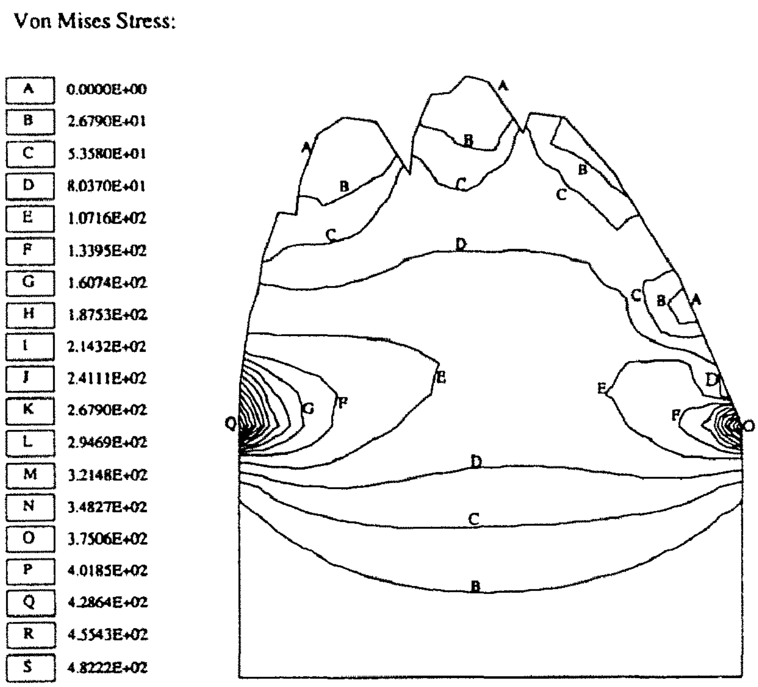
Fig. 3. Calculation of the plate stresses (*continued overleaf*).

Moreover the estimate (64) holds for the present problem under the mild assumptions of the preservation of Markov's inequality and of the k-unisolvability.

(ii) As a second example, let us consider the plate bending problem of Fig. 2. The plate is linear elastic with $E = 2,1 \times 10^6\ \mathrm{t\,m^{-2}}$, Poisson ratio $\nu = 0.16$ and thickness $t = 15$ cm. The boundary part F of the structure is a fractal which is approximated by the boundary sequence $F_0, F_1, F_2, \ldots, F_5$, according to (14). For each F_i ($i = 2, 3, 4, 5$) the deflexions of the plate and the von Mises stresses are depicted in Figs 2a–d and 3a–d. After the third



c)



d)

Fig. 3 *continued*

boundary approximation the differences in the stress and the deflections become insignificant as was expected from proposition 1. Random approximations of the fractal boundary give worse results, as it is also theoretically obvious: indeed a random choice of the interface only by chance could give a better result than the fixed point algorithm generated by the operator T.

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